

INDEX

A

Abel, Niels Henrik, 31
abstract algebra
 abelian, 31
 Cayley table, 33–34
 groups, 31
 cyclic, 33
 isomorphic, 34
 Klein four-group, 33
 magma, 31
 monoid, 31
algebraic number, 22
Al-Karaji, 207
Animals game, 262
Anscombe, Francis, 330
arithmetic series, 120
automatic differentiation, 402, 405

B

backpropagation, 413
Backus, John, 268
Bayes' theorem, 303–305
Bessel, Friedrich, 317
binary adder, 67–69
binary algebra, 48
binary numbers, 3–4
binary relations
 Cartesian product, 78–79, 85
 in code, 87–89
 complement, 85
 composition, 82–84
 converse, 85
 defined, 79–80
 equivalence classes, 90–91
 equivalence relations, 89–90
 infinite sets, 82
 as matrix, 81
 n -ary relations, 86

binary search, 112–114. *See also* trees:
 binary search tree

binomial theorem, 208

BLAS, 368

Boolean algebra

 binary, 48–49

 definition, 44

 functions, 51–55

 laws, 44

 propositional logic, 46–48

 set algebra, 45

 truth tables, 46

Boolean functions, 51

 algebraic simplification, 61–63

 canonical normal forms, 55

 in code, 54–55

 complement, 54

 as digital circuits, 65

 duals, 52

 Karnaugh maps, 57–61

 product of sums (POS), 56–57

 sum of products (SOP), 56

Boute, Raymond T., 165

Box, George, 462

box plots

 interquartile range (IQR), 322–323

 whiskers (fliers), 322

bubble sort, 111–112

butterfly effect, 463

C

calculus. *See* differential calculus;
 integral calculus

Cantor, Georg, 34

Cantor's diagonal proof, 38

Cartesian product, 78–79, 85

 cardinality, 81

Cayley table, 33–34

central limit theorem (CLT), 299, 318

- chicken nuggets problem, 104–105
 - Clifford, William, 405
 - Cohen, Paul, 39
 - Cohen's d , 335, 338
 - Collatz sequence, 122–124
 - combinatorics, 187
 - combinations, 203–207
 - permutations, 199–203
 - pigeonhole principle, 196–199
 - probabilistic, 201
 - principle of inclusion-exclusion (PIE), 195
 - principles of counting, 188–189
 - product rule, 193
 - sum rule, 192
 - complex numbers, 21, 127–128
 - composite numbers, 105, 152
 - computable numbers, 39–40
 - confidence interval (CI), 338–342
 - independent, 339
 - paired, 339
 - congruences, 166–169
 - classes, 168
 - properties, 167
 - continuous distributions, 294–299
 - continuum hypothesis, 38–39
 - correlation, 324
 - Pearson, 324–327
 - Spearman, 327–329
 - countably infinite set, 35
 - covariance, 325
 - cross product, 354–355
 - cyclic group, 33
- D**
- darts metaphor (for estimating π), 425–426
 - decision problem, 39
 - derangements, 183, 199
 - differential calculus
 - automatic differentiation, 402, 405
 - derivatives, 384–394
 - first, 386
 - second, 386, 397
 - extrema, 395
 - locating with Newton's method, 397–399
 - first derivative test, 396
 - gradient descent, 409–413
 - inflection point, 396–397
 - notation, 386
 - numeric differentiation, 402, 403
 - partial derivatives, 399–402
 - chain rule, 401–402
 - mixed, 400–401
 - rules
 - chain, 391–392, 401–402
 - constant, 387
 - exponentials, 393
 - hyperbolic trigonometric functions, 393
 - logarithms, 394
 - power, 387–388
 - product, 388–389
 - quotient, 389–490
 - sum, 387–388
 - summary chart, 394–395
 - trigonometric, 390
 - secant line, 385
 - second derivative test, 397
 - slope, 384–385
 - symbolic differentiation, 408–409
 - tangent line, 384–385
 - differential equation
 - first-order, 436–437
 - numerical solutions, 439
 - Euler's method, 440–441
 - Runge–Kutta method, 441–444
 - ordinary, 436–439
 - projectile motion with drag, 452–456
 - second-order, 438–439
 - simple pendulum, 444
 - damping, 450–451
 - SIRD model, 460–462
 - SIR model, 456–460
 - digital circuits, 64
 - binary adder, 67–69
 - combinational, 64
 - logic gates, 64, 65, 67, 69
 - digital logic, 49
 - truth tables, 50
 - Dijkstra, Edsger, 230

Diophantine equations, 174
 Erdős–Straus conjecture, 176–177
 linear, 174–176
 nonlinear, 176–177
Diophantus of Alexandria, 174
distributions. *See* probability: distributions
divisibility properties, 160
dot product, 352–353
dual numbers, 405–408

E

effect size, 335
Elonen, Jarno, 366
encryption, 178
endianness, 11–112
Ensemble Klang, 182
equivalence classes, 90–91
equivalence relations, 89–90
 Hasse diagrams, 93
 number sets, 91–92
 partial orderings, 92–93
Eratosthenes of Cyrene, 154
Erdős, Paul, 123
Euclid–Mullin sequence, 153
Euler’s method, 440–441
Euler’s totient function, 177–178

F

Fibonacci sequence, 101, 119, 135, 143
Fisher, Ronald, 334
floating-point, 2, 12, 14
 infinity, 16
 NaN (not a number), 16
 round-off error, 14, 15–16
 significand, 13
 spacing, 15
functions
 composition, 77–78
 definition, 74
 types, 76
fundamental theorem of arithmetic, 105
fundamental theorem of calculus, 424

G

Gaussian integers, 27
geometric series, 120
Germain, Sophie, 156
gnome sort, 114–115

Gödel, Kurt, 39
Goldbach conjecture, 159
Gosset, William Sealy, 340
gradient descent, 409–413
 optimization via, 410
graphs
 adjacency lists, 216
 adjacency matrices, 217–219
 breadth-first traversal, 219–222
 clique, 214
 complete, 214
 concepts, 213
 depth-first traversal, 222–225
 Dijkstra’s algorithm, 230–233
 directed, 212–213
 directed acyclic graph (DAG),
 233–236
 isomorphism, 214–215
 planar, 215
 searching, 225–227
 terminology, 212
 topological sort, 234, 235–236
 undirected, 212–213
 unweighted shortest path, 228–229
greatest common divisor (gcd), 161
 Bézout’s identity, 171
 Euclidean algorithm, 162
 extended, 171
 properties, 164
groups, 31
 cyclic, 33
 isomorphic, 34
Guinness Brewery, 340
Gupta, Shyam Sunder, 184
Guy, Richard, 184

H

Hadamard product, 354
halting problem, 39
Hamiltonian numbers, 21
Hasse, Helmut, 93
Helfgott, Harald, 159
hexadecimal numbers, 4
Hilbert, David, 19, 34
histograms, 286–287
Hopfield network, 362
Horner’s method, 7
hypothesis testing, 331

I

IEEE 754, 12, 17, 297, 367
IFS (iterated function system), 377
inclusion-exclusion, principle (PIE), 195
induction
 base case, 96
 inductive hypothesis, 96
 loop invariant, 109–110
 predicate function, 96
 strong, 103–105
 weak, 96
infinite sets
 Cantor’s diagonal proof, 38
 cardinality, 34–35, 38
 continuum hypothesis, 38–39
 countably infinite, 35
 denumerable, 35
 power sets, 38
 uncountably infinite, 36, 38
inner product, 352
integers, 2, 21
 endianness, 11–12
 storage formats, 8
 two’s complement, 9–10
 unsigned, 9
integer sequences, 177
 aliquot sequence, 179–180
 aliquot sum, 178–179
 amicable numbers, 180
 aspiring numbers, 180
 deficient numbers, 180
 Euler’s totient function, 177–178
 harshad numbers, 180–182
 Narayana’s cows, 182
 perfect numbers, 179–180
 proper divisors, 178–179
 Ruth–Aaron numbers, 184
 Smith numbers, 183–184
 sociable numbers, 180
integral calculus
 adaptive numerical integration,
 431–433
 antiderivatives, 418
 with darts, 425–426
 definite integrals, 417
 dummy variables, 419
 fundamental theorem of
 calculus, 424

indefinite integral, 418–420
integrand, 417
integration, 417
 by partial fractions, 423
 by parts, 422
 substitution, 421
Monte Carlo integration, 427
Riemann sum, 416
Simpson’s rule, 429, 433
trapezoidal rule, 428–429
interval data, 309
irrational numbers, 22
isomorphic graphs, 214–215
isomorphic groups, 34
iterated function system (IFS), 377

J

Johnson, Tom, 182

K

Karnaugh, Maurice, 57
Karnaugh maps, 57–61
Kermack, W.O., 456
Klein four-group, 33
Kaprekar, D.R., 180
Kronecker, Leopold, 34

L

LAPACK, 368
Laplacian matrix, 243–244
Las Vegas algorithms, 427
law of large numbers (LLN), 299–302
least common multiple (lcm), 160–164
 properties, 164
Leibniz, Gottfried, 384
lightning strikes, 205
linear algebra. *See also* matrices; vectors;
 vector spaces
 affine transformations, 376
 cross product, 354–355
 definitions, 346
 dot product, 352–353
 Gauss–Jordan elimination, 365–366
 Hadamard product, 354
 inner product, 352
 linear equations
 augmented matrix, 365
 consistency, 364

- linear variables, 364
- row operations, 365
- systems of, 364
- linearly (in)dependent, 357–359
- linear transformations, 374
- linear vector equations, 366–368
- matrix multiplication, 360
- outer product, 362
- parallelogram rule, 347–348
- vector sum (resultant), 348
- logic gates, 64, 65, 67, 69
- logistic map, 125–127
- loop invariant, 109–110
- Lorenz, Edward, 462
- Lorenz attractor, 462–466
- Lucas, Édouard, 148
- Lucas sequence, 119, 136

M

- Mandelbrot, Benoit, 127
- Mandelbrot set, 127–129
- Mann–Whitney U test, 332
- matrices, 359
 - characteristic equation, 379
 - column space, 370
 - determinants
 - 2×2, 372
 - 3×3, 373
 - $n \times n$, 373
 - properties, 372
 - eigenvalues, 378
 - eigenvectors, 378
 - identity, 367
 - inverse, 367–368
 - multiplication, 360–362
 - formula, 360
 - properties, 360
 - rank, 370–371
 - rotation, 374–376
 - row space, 370–371
 - singular, 367
 - square, 369–374
 - properties, 369–370
 - symmetric, 369
 - trace, 369
 - triangular, 372
 - zero determinant, 372

- McKendrick, A.G., 456
- mean value theorem for integrals, 427
- Mersenne primes, 156, 180
- Millennium Prize, 416
- Miller–Rabin primality test, 158
- Milton, John, 156
- modular arithmetic, 165
 - congruences, 166–169
 - equations, 169–174
 - inverses, 169–171
 - linear equations, 173–174
- Monte Carlo algorithm, 427
- Monte Carlo integration, 427
- Mullin, Albert, 153
- mystic roses (graphs), 214

N

- natural numbers, 20
- Newton, Isaac, 384
- Newton’s method, 397–399
- nominal data, 308
- norm (magnitude), 352
- normal (continuous) distributions, 296
- number base conversion, 5–8
- number types, 22, 27
 - computable numbers, 39–40
 - dual numbers, 405–408
 - floating-point, 2
 - integer, 2
 - natural numbers, 20
 - in Python, 29
 - real, 2
 - relationships, 27
 - uncomputable numbers, 40–42
- numeric differentiation, 402, 403
- NumPy library, 13

O

- octal numbers, 4, 6
- octonion, 21, 30
- OEIS (Online Encyclopedia of Integer Sequences), 104
 - amicable numbers (IA063990), 180
 - aspiring numbers (A063769), 180
 - balanced primes (A006502), 156
 - chicken nuggets (A065003), 104–105
 - deficient numbers (A191363), 180

- OEIS (*continued*)
- Euclid–Mullin (A000945), 153
 - Fibonacci primes (A001605), 156
 - harshad numbers (A005349), 181
 - Lucas pseudoprimes (A005845), 119
 - Mersenne primes (A000043), 156
 - Narayana’s cows (A000930), 182
 - perfect numbers (A000396), 180
 - Ruth–Aaron numbers
 - (A006145), 184
 - sequence audio, 184
 - sexy primes (A023201), 155
 - Smith numbers (A006753), 184
 - Sophie Germain primes
 - (A005384), 156
 - subfactorials (A000166), 183
 - twin primes (A077800), 155
 - twin primes, form 2 (A071695), 155
 - one-hot vectors, 308
 - optimization via gradient descent, 410
 - ordinal data, 308–309
 - outer product, 362
- P**
- Pandita, Narayana, 182
 - partial orderings, 92–93
 - Pascal, Blaise, 207
 - Pascal’s triangle, 207–209
 - pigeonhole principle, 196–199
 - Poisson distribution, 292
 - Pomerance, Carl, 184
 - POS (product of sums), 56–57
 - prime factorization, 105
 - prime numbers, 152
 - balanced primes, 156
 - Belphegor’s prime, 156
 - Euclid–Mullin sequence, 153
 - Euclid’s theorem, 152
 - Fibonacci primes, 155–156
 - Goldbach conjecture, 159
 - Jenny’s prime, 155
 - Mersenne primes, 156, 180
 - Miller–Rabin primality test, 158
 - relatively prime (coprime), 163
 - right-truncatable primes, 8, 156
 - semiprimes, 158
 - sexy primes, 155
 - sieve of Eratosthenes, 154–155
 - Sophie Germain primes, 156
 - twin primes, 155
 - Prim’s algorithm, 245
 - probability
 - Bayes’ theorem, 303–305
 - evidence, 303
 - likelihood, 303
 - prior, 303
 - Box–Muller transformation, 297
 - central limit theorem (CLT),
 - 299, 318
 - chain rule, 283–285
 - concepts, 269
 - conditional, 275–277
 - contingency table, 280
 - definition, 268
 - distributions, 285
 - arbitrary discrete, 293
 - Bernoulli, 291–292
 - beta (continuous), 296
 - binomial, 289
 - gamma (continuous), 296
 - normal (continuous), 296
 - Poisson, 292
 - sampling (continuous), 296–299
 - uniform (continuous), 297
 - uniform (discrete), 298
 - events, 270
 - dependent, 276
 - independent, 272
 - false-positive rate, 303
 - histograms, 286–287
 - joint, 272, 279–283
 - law of large numbers (LLN),
 - 299–302
 - marginal, 279, 281–283
 - negative predictive value (NPV), 304
 - normalizing, 293
 - Poisson distribution, 292
 - population mean, 299–302
 - positive predictive value (PPV), 304
 - posterior, 303
 - prevalence, 303
 - probability density function (PDF),
 - 294–295
 - probability mass function
 - (PMF), 290
 - product rule, 272–273

- updated, 276
- random variables
 - continuous, 268–269
 - discrete, 268–269
- sensitivity, 303
- sequential search, 293–294
- sum rule, 271
 - updated, 274–275
- total, 277–279
- product of sums (POS), 56–57
- propositional logic, 46–48
- pseudorandom generator, 121–122
- Pythagorean triples, 155

Q

- quaternion, 21
- Quicksort, 146–145

R

- radix point, 3
- randomized algorithms, 427
- ratio data, 309
- rational arithmetic, 15
- rational numbers, 21
- real numbers, 2, 21–22
- recurrence relations, 7, 118
 - characteristic polynomials, 130
 - first-order, 129–134
 - linear, 119–122
 - nonlinear, 125–129
 - second-order, 134–139
 - solving, 129–139
- recursion, 139
- Riemann, Bernhard, 416
- Riemann sum, 416
- Rivest–Shamir–Adleman (RSA)
 - encryption, 178
- RMS *Titanic*, 281
- round-off error, 14, 15–16
- Russell, Bertrand, 96

S

- Sarbazi-Azad, Hamid, 114
- Schrödinger equation, 378
- SciPy library, 124, 299
- sedenion, 21, 30
- set algebra, 45
- set builder notation, 21

sets

- algebra, 45
- cardinality, 20
- definition, 20
- elements of, 20–21
- empty sets, 21
- laws, 28–29
- operations, 22–24
 - difference, 23
 - intersection, 23
 - symmetric difference, 23
 - union, 22–23
- power sets, 28
- proper subsets, 26
- set builder notation, 21
- subsets, 26
- supersets, 26
- Venn diagrams, 25–26
- sexagesimal numbers, 3
- Sierpiński triangle, 209, 377
- sieve of Eratosthenes, 154–155
- SIRD model, 460–462
- SIR model, 456–460
- Sorenson, Jonathan, 155
- standard deviation, 317
- statistics, 307
 - alternative hypothesis, 331
 - Anscombe’s quartet, 330
 - Bessel’s correction, 317
 - box plots
 - interquartile range (IQR), 322–323
 - whiskers (fliers), 322
 - Cohen’s *d*, 335, 338
 - confidence interval (CI), 338–342
 - independent, 339
 - paired, 339
- correlation, 324
 - Pearson, 324–327
 - Spearman, 327–329
- covariance, 325
- covariates, 331
- degrees of freedom, 339
- descriptive, 310
- effect size, 335
 - paired, 338
- hypothesis testing, 331

statistics (*continued*)

- independent samples, 331–335
- interval data, 309
- Mann–Whitney U test, 332
- mean
 - arithmetic, 310
 - geometric, 311
 - harmonic, 312
 - power, 312
- measures of central tendency, 310
- measures of variation, 310, 316–319
- median, 314
- median absolute deviation (MAD), 317
- mode, 315
- nominal (categorical) data, 308
- nonparametric test, 327
- null hypothesis, 331, 333–335, 338
- ordinal data, 308–309
- outliers, 310
- parametric test, 327
- percentiles, 320
- p-values, 332
- quantiles, 320
- quartiles, 320
- range, 316
- ratio data, 309
- root mean square (RMS), 312
- standard deviation, 317
- standard error of the mean, 317
- statistically significant, 334
- suggested approach, 340
- summary, 310
- t-test (independent), 332
- t-test (paired), 336
- variance (biased), 316
- variance (unbiased), 317
- Welch’s t-test, 332
- Wilcoxon rank-sum test, 333
- Wilcoxon signed-rank test, 336

Straus, Ernst, 177

Strogatz, Steven, 463

subfactorials, 183

sum of products (SOP), 56

Sweigart, Al, 178

symbolic differentiation, 408–409

SymPy, 124, 408–409, 430–431

T

Taylor series, 403

Towers of Hanoi, 148

transcendental numbers, 22

transistor logic gates, 69–70

trees

- abstract syntax tree (AST), 259
- Animals* game, 262
- binary search tree (BST), 247–248, 249–253
 - in code, 254–259
 - searching, 253
 - traversal, 249–252
- Cayley’s formula, 241
- definition, 240
- forest, 240
- matrix-tree theorem, 243
- minimum spanning tree (MST), 244–246
- rooted, 246
- spanning tree, 241
 - locating, 242
- subtree, 246
- traversal
 - breadth-first, 249–250
 - in-order, 251
 - post-order, 252
 - pre-order, 250–251

truth tables, 46

t-test

- independent, 332
- paired, 336

Turing, Alan, 39

Turing machines, 39–40

- decision problem, 39
- halting problem, 39

Tutone, Tommy, 155

two’s complement, 9–10

U

uncomputable numbers, 40–42

uncountably infinite set, 36, 38

undecidable (hypothesis), 39

uniform (continuous) distributions, 297

universal product codes (UPCs), 169

V

vectors

- column, 350–351
- cross product, 354–355
- definition, 346
- dot product, 352–353
- geometric, 347–348
- inner product, 352
- norm (magnitude), 352
- normal, 355
- notation, 350
- orthogonal, 352
- orthonormal, 352
- outer product, 362
- parallelogram rule, 347–348
- projection, 353
- row, 350–351
- sum (resultant), 348
- transpose, 350
- tuples, 349

vector spaces, 346

- basis, 358
- dimensionality, 358
- standard basis, 358
- subspaces, 357

Venn, John, 25

Venn diagrams, 24–26

von Neumann, John, 21, 272

W

Wauwatosa, Wisconsin, 205–206

Wigginton, Randy, 261

Wilansky, Albert, 183

Wilcoxon rank-sum test, 333

Z

Zermelo–Fraenkel (ZFC) set theory, 39